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A study of solving Unconstrained Geometric Programming Problem and Its Application

Abstract:

Constrained Geometric Program is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. In this article, a unconstrained Geometric Programming problem is defined in usual form with unrestricted exponents a_{ij} i=1,2,3,….n and j=1,2,3,….N ($n \leq N+1$). An attempt is made to obtain an optimum solution of this model using derivative and matrix inversion method. An example is considered to illustrate the procedure.

Key Words:

Unconstrained Geometric Programming problem, Convex Optimization method, Primal and Dual problem, Orthogonality and normality conditions,

Introduction:

Duffin, Peterson, and Zener [2] published a book "*Geometric Programming: Theory and Applications"* that started the field of Geometric Programming as a branch of nonlinear optimization with many useful theoretical and computational properties of Geometric Programming, to a large extent the scope of linear programming applications and is naturally applied to several important nonlinear systems in science and engineering. Several important developments of Geometric Programming are in the area of mechanical and civil engineering, chemical engineering, probability and statistics, finance and economics, control theory, circuit design, information technology, coding and signal processing, wireless networking, etc. took place in the late 1960s and 1970s. There are several books on nonlinear optimization that have a section on Geometric Programming, e.g., M. Avriel, [5], C. S. Beightler [1], G. Hadley [4], Taha [6], etc. However, many researchers felt that most of the theoretical, algorithmic and application aspects of Geometric Programming had been exhausted by the early 1980's, the period of 1980–98 was relatively quiet. After the revolution in the electronic field, over the last few years, Geometric Programming started to receive renewed attention from the operations research community.

$f(x)$ R. Duffin and C. Zener [3], have defined unconstrained Geometric Programming in the following manner:

$$Z = f(\underline{x}) = \sum_{j=1}^{N} u_j$$

Where, [1]

$$u_j = c_j \prod_{i=1}^{n} x_i^{a_{ij}} \quad for \quad j = 1, 2, 3, ..., N$$

Here it is assumed that the coefficient $c_j > 0$ and N is finite. The exponents a_{ij} are unrestricted in sign i.e. it may be positive or negative. The function $f(\underline{x})$ takes the form of a polynomial, except that the exponents a_{ij} may be negative.

Mathematical Procedure:

Consider the Geometric Programming problem as:

$$Minimise \quad Z = f(\underline{x}) = \sum_{j=1}^{N} u_j$$

Where,

$$u_j = c_j \prod_{i=1}^{n} x_i^{a_{ij}} \quad for \quad j = 1, 2, 3, ..., N$$

[2]

This problem will be considered as the primal problem. Here $f(\underline{x})$ is in the polynomial form and it is assumed that all variables x_i are strictly positive so that the region for which $x_i < 0$ represents the infeasible solution space. The requirement $x_i \neq 0$ plays an essential role in the derivation of the results.

For minimum value of the objective function, the first order partial derivative of z must be zero, now differentiate z with respect to x_k (k=1,2,3,.....,n)

$$\frac{\partial z}{\partial x_k} = \sum_{j=1}^{N} \frac{\partial u_j}{\partial x_k}$$
$$= \sum_{j=1}^{N} c_j \cdot a_{ij} (x_i^{'})^{a_{ij}-1} \cdot \prod_{i \neq k}^{n} x_i^{a_{ij}} = 0, \quad for \quad k = 1, 2, 3, ..., n$$

[3]

Since, each $x_i > 0$

$$\frac{\partial z}{\partial x_k} = 0 = \frac{1}{x_k} \sum_{j=1}^{N} a_{ij} u_j, \quad for \quad k = 1, 2, 3, ..., n$$

[4]

Let z^* be the minimum value of z. It can be easily seen that $z^* > 0,$ since each $x_k^* > 0$ and z is a polynomial defined as

$$y_j = \frac{u_j^*}{z^*}$$ [5]

Which shows that $y_j > 0$ and

$$\sum_{j=1}^{N} y_j = 1 \qquad \left(\because \sum_{j=1}^{N} y_j^* = z^* \right)$$ [6]

Thus the value of y_j represents the relative combination of the jth term u_j to the optimum value of the objective function z^*.

Now the necessary conditions can be written as

$$\sum_{j=1}^{N} a_{kj} \cdot y_j = 0 \quad (z^* > 0,\ x_k > 0,\ k = 1,2,3,....n)$$ [7]

and

$$\sum_{j=1}^{N} y_j = 1, \qquad (y_j > 0,\ j = 1,2,3,....N)$$ [8]

These conditions [7] and [8] are known as orthogonality and normality conditions. By using matrix inversion method, these conditions will give a unique solution for y_j, if (n+1) = N and all the equations are independent. If N > (n+1) then the problem becomes more complex because the values of y_j are not unique. However, it is possible to determine y_j uniquely for the purpose of minimizing z.

Now, suppose that y_j^* are the unique values determined from the equations given in the results [7] and [8]. These values are used to determined the values of z^* and x_k^* for $i = 1,2,3,....n$ as under,

Consider,

$$z^* = \left(z^*\right)^{\sum_{j=1}^{N} y_j^*} \quad \left(\because \sum_{j=1}^{N} y_j^* = 1 \right)$$ [9]

$$z^* = \prod_{j=1}^{N} \left(\frac{c_j \cdot \prod_{i=1}^{n} x_i^{a_{ij}}}{y_j^*} \right)^{y_j^*}$$ [10]

$$z^* = \prod_{j=1}^{N}\left(\frac{c_j}{y_j^*}\right)^{y_j^*}\left[\prod_{j=1}^{N}\left(\prod_{i=1}^{n} x_i^{a_{ij}}\right)^{y_j^*}\right]$$ [11]

$$z^* = \prod_{j=1}^{N}\left(\frac{c_j}{y_j^*}\right)^{y_j^*}\left[\prod_{i=1}^{n} x_i^{\sum_{j=1}^{N} a_{ij}\cdot y_j^*}\right]$$ [12]

$$z^* = \prod_{j=1}^{N}\left(\frac{c_j}{y_j^*}\right)^{y_j^*} \qquad \left(\because \sum_{j=1}^{N} a_{ij}\cdot y_j^* = 0\right)$$ [13]

Thus, the value of z^* is determined from result [13] as soon as all y_j^* are determined.

Now, for known values of y_j^* and z^* the value of u_j^* can be determined from $u_j^* = y_j^* \cdot z^*$

Since $u_j^* = c_j \prod_{i=1}^{n}\left(x_i^*\right)^{a_{ij}} \quad for \quad j = 1, 2, 3, ..., N$ simultaneously solution of these equations should give $x_j^* \ for \ i = 1, 2, 3, ...n.$

The procedure described hare shows that the solution to the original polynomial z can be transformed in to the solution of a set of linear equations in y_j. Observed that all y_j^* are determined from the necessary conditions for a minimum. However , it can be shown that, these conditions are also sufficient.

Notes:

The proof under the given restriction on z is given in Beightler [1].

The variables y_j actually defined as the dual variables associated with the primal problem. These relationship can be explained as under.

$$z = \sum_{j=1}^{N} y_j\left(\frac{u_j}{y_j}\right)$$ [14]

Now, consider the following function

$$w = \prod_{j=1}^{N}\left(\frac{u_j}{y_j}\right)^{y_j}$$ [15]

$$w = \prod_{j=1}^{N} \left(\frac{c_j \prod_{i=1}^{n} x_i^{a_{ij}}}{y_j} \right)^{y_j} \qquad [16]$$

$$w = \prod_{j=1}^{N} \left(\frac{c_j}{y_j} \right)^{y_j} \qquad [17]$$

Since, $\sum_{j=1}^{N} y_j = 1$ and $y_j > 0$ by using Cauchy's inequality, it can be said that $w \le z$.

The function w with its variables $y_1, y_2, y_3, \dots\dots, y_N$, defined as the dual to the primal problem. Since w represents the lower bound on z and since z is associated with the minimization problem, it follows by maximizing w that

$$w^* = \underset{y_j}{Max}\; w = \underset{x_j}{Min}\; z = z^* \qquad [18]$$

This means that the maximum value of $w = w^*$ over the values of $y_j,\ \ j = 1, 2, 3, \dots\dots N$ is equal to the minimum values of $z = z^*$ over the values of $x_i,\ \ i = 1, 2, 3, \dots\dots n$

Application to Hypothetical Problem:

Consider the following problem of Geometric Programming;

$$Minimise\; z = \frac{1}{x_1 x_2 x_3} + 2x_2 x_3 + 3x_1 x_3 + 4x_1 x_2$$

subject to the condition that all variables have positive values i.e. $x_1, x_2, x_3 > 0$

For solving the above problem let us first consider the given function as,

$$\begin{aligned} Minimise\; z &= x_1^{-1} \cdot x_2^{-1} \cdot x_3^{-1} + 2x_1^{0} \cdot x_2^{1} \cdot x_3^{1} + 3x_1^{1} \cdot x_2^{0} \cdot x_3^{1} + 4x_1^{1} \cdot x_2^{1} \cdot x_3^{0} \\ &= u_1 + u_2 + u_3 + u_4 \\ &= c_1 \cdot x_1^{a_{11}} \cdot x_2^{a_{21}} \cdot x_3^{a_{31}} + c_2 \cdot x_1^{a_{12}} \cdot x_2^{a_{22}} \cdot x_3^{a_{32}} + c_3 \cdot x_1^{a_{13}} \cdot x_2^{a_{23}} \cdot x_3^{a_{33}} + c_4 \cdot x_1^{a_{14}} \cdot x_2^{a_{24}} \cdot x_3^{a_{34}} \end{aligned}$$

Then by comparison, following matrices are obtained.

$[c_1\ c_2\ c_3\ c_4] = [1\ 2\ 3\ 4]$ and $\begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 01 \end{pmatrix}$

Here $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ so the case in which N = (n+1) is to be considered. Using orthogonality and normality conditions

$$\sum_{j=1}^{4} a_{kj} \cdot y_j = 0 \quad (z^* > 0,\ x_k > 0,\ k = 1, 2, 3)$$ and $$\sum_{j=1}^{4} y_j = 1, \quad (y_j > 0)$$

Following equations are obtained

$$\begin{aligned} &a_{11} \cdot y_1 + a_{12} \cdot y_2 + a_{13} \cdot y_3 + a_{14} \cdot y_4 = 0 \\ &a_{21} \cdot y_1 + a_{22} \cdot y_2 + a_{23} \cdot y_3 + a_{24} \cdot y_4 = 0 \\ &a_{31} \cdot y_1 + a_{32} \cdot y_2 + a_{33} \cdot y_3 + a_{34} \cdot y_4 = 0 \\ &and \qquad\qquad y_1 + y_2 + y_3 + y_4 = 1 \end{aligned}$$

The above equations can be represented as

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, by using matrix inversion method,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -0.2 & -0.2 & -0.2 & 0.4 \\ -0.6 & 0.4 & 0.4 & 0.2 \\ 0.4 & -0.6 & 0.4 & 0.2 \\ 0.4 & 0.4 & -0.6 & 0.2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$$

This is a n unique solution given as

$$y_1^* = o.4, \quad y_2^* = o.2, \quad y_3^* = o.2, \quad y_4^* = o.2.$$

Now,

$$z^* = \prod_{j=1}^{4} \left(\frac{c_j}{y_j^*} \right)^{y_j^*}$$
$$= \left(\frac{1}{0.4} \right)^{0.4} \cdot \left(\frac{2}{0.2} \right)^{0.2} \cdot \left(\frac{3}{0.2} \right)^{0.2} \cdot \left(\frac{1}{0.2} \right)^{0.2} \cdot$$
$$= 7.803$$

From the equation $u_j^* = y_j^* \cdot z^*$ it can be deduced that,

$$u_1 = 2.8332, \; u_2 = 1.4166, \; u_3 = 1.4166, \; u_4 = 1.4166,$$

Which will gives the optimum solution to the primal problem as under

$$x_1^* = 0.498, \; x_2^* = 0.747, \; x_3^* = 0.948, \; and \quad z^* = 7.083.$$

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Mr. Hiren S. Doshi
Associate Professor
H L Collge of Commerce
Navrangpura, Ahmedabad